Signal Analysis & Communication ECE355

Ch. 4.3: Properties of CTFT

Lecture 20

25-10-2023

Ch. 4.3: PROPERTIES OF CTFT

- Similar to CTFS properties, the CTFT properties provide us with a significant amount of insight into the transform & into the relationship b/w the time-domain & freq.-domain description of the sig.
- Moreover, many of the properties are often useful in reducing the complexity of the evaluation of Fourier Transform or inverse Fourier Transform.
- Also, recall $\quad CTFS \xrightarrow{T \to \infty} CTFT$

$$\text{CTFT of Periodic } x_{(t)} = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

CTFS coefficients.

- Notation: $\quad x_{(t)} \overset{\mathcal{F}}{\longleftrightarrow} X_{(j\omega)}$

① Linearity:

If $\quad y_{(t)} \overset{\mathcal{F}}{\longleftrightarrow} Y_{(j\omega)}$

$$a x_{(t)} + b y_{(t)} \overset{\mathcal{F}}{\longleftrightarrow} a X_{(j\omega)} + b Y_{(j\omega)}$$

② Time Shifting

$$x_{(t-t_0)} \overset{\mathcal{F}}{\longleftrightarrow} X_{(j\omega)} e^{-j\omega t_0}$$

Proof:

LHS $\quad x_{(t-t_0)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{(j\omega)} e^{j\omega(t-t_0)} d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(e^{-j\omega t_0} X_{(j\omega)}\right)} e^{j\omega t} d\omega$$

③ Time & Freq. Scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Proof:

$$\text{CTFT of } x(at) = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$\overset{\tau = at}{=} \begin{cases} \displaystyle\int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau \cdot \left(\frac{1}{a}\right) & , a>0 \\ \displaystyle\int_{\infty}^{-\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau \cdot \left(\frac{1}{a}\right) & , a<0 \end{cases}$$

$$= \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

- Special Case

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega) \qquad (\text{when } a=-1)$$

④ Conjugation

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

Proof: Similar to CTFS case.

Special Cases

I. $x(t)$ is Real

$$x(t) = x^*(t)$$

$$X(j\omega) = X^*(-j\omega)$$

II. $x(t)$ is Real & Even

$X(j\omega)$ is Real & Even.

III. $x(t)$ is Real & Odd

$X(j\omega)$ is purely Imaginary & Odd.

Recall:

- Every $x(t)$ can be expressed as a sum of even & odd functions.

$$x(t) = x_e(t) + x_o(t)$$

- If $x(t)$ is Real.

Via linearity property ①

$$\mathcal{F}\{x(t)\} = \underbrace{\mathcal{F}\{x_e(t)\}}_{\text{Real}} + \underbrace{\mathcal{F}\{x_o(t)\}}_{\text{Purely Imaginary}}$$

$$\Rightarrow x_e(t) \overset{\mathcal{F}}{\longleftrightarrow} Re\{X(j\omega)\}$$

$$x_o(t) \longleftrightarrow j\,Im\{X(j\omega)\}$$

& $$X(j\omega) = Re\{X(j\omega)\} + j\,Im\{X(j\omega)\}$$

Example

$$x(t) = e^{-a|t|}, \quad a>0 \; ; \; X(j\omega) = ?$$

$$x_{(t)} = e^{-at}u_{(t)} + e^{at}u_{(-t)}$$

$$= 2\,Ev.\{e^{-at}u_{(t)}\}$$

Recall: $Ev.\{x_{(t)}\} = \frac{x_{(t)} + x_{(-t)}}{2}$

$$X_{(j\omega)} = 2\,Re\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

⑤ Differentiation & Integration

(i) $x'_{(t)} \longleftrightarrow j\omega X_{(j\omega)}$

Proof

$$x'_{(t)} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_{(j\omega)}\, e^{j\omega t}\, d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X_{(j\omega)}\cdot(j\omega)}\, e^{j\omega t}\, d\omega$$

(ii) $\int_{-\infty}^{t} x_{(\tau)}\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X_{(j\omega)} + \pi X_{(0)}\delta_{(\omega)}$

Proof:

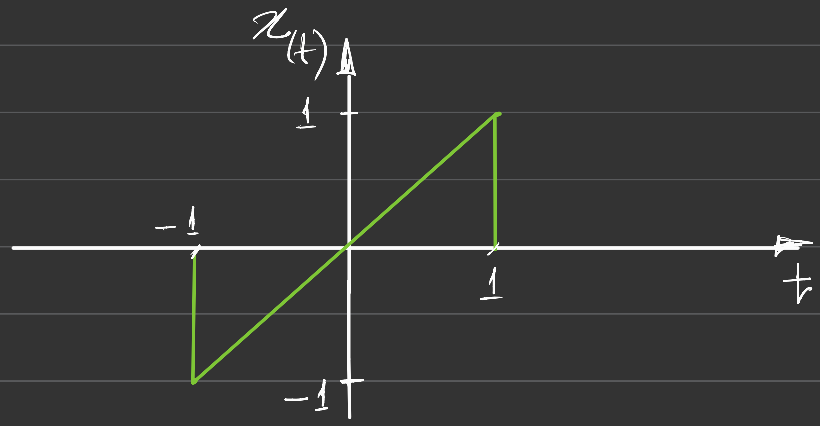
$$\int_{-\infty}^{t} x_{(\tau)}\, d\tau = x_{(t)} * u_{(t)}$$

$$\overset{\mathcal{F}}{\longleftrightarrow} X_{(j\omega)}\, U_{(j\omega)}$$

$$= X_{(j\omega)}\left[\frac{1}{j\omega} + \pi\delta_{(\omega)}\right]$$

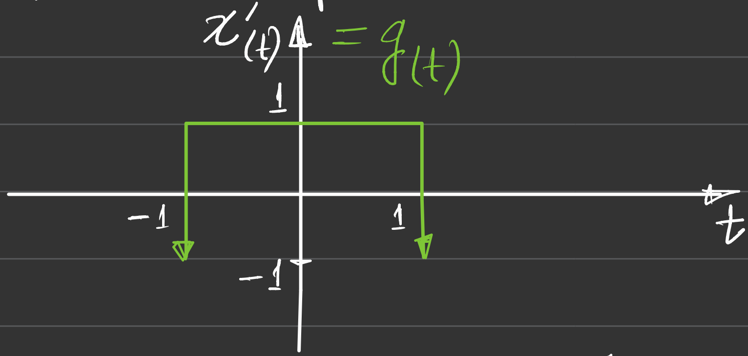
$$= \frac{1}{j\omega} X_{(j\omega)} + \pi X_{(0)}\delta_{(\omega)}$$

Example



$$X_{(j\omega)} = ?$$

- Considering the differential of $x_{(t)}$, which gives us a pulse & two impulses at the points of discontinuities. And, we know the FT of both, pulse & impulse.



- $$G_{(j\omega)} = \underbrace{\frac{2\sin\omega}{\omega}}_{\text{CTFT of pulse}} - \underbrace{e^{j\omega} - e^{-j\omega}}_{\text{CTFT of shifted impulses}}$$

$$= \frac{2\sin\omega}{\omega} - 2\cos\omega$$

- Now via integration property : $\left[x_{(t)} = \int_{-\infty}^{t} g_{(\tau)}\, d\tau\right]$

$$X_{(j\omega)} = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega} + \pi G_{(0)}\,\delta_{(\omega)}$$

$$\underbrace{G_{(0)} = 2 - 2 = 0}_{= 0}$$

⑥ Duality:

- Before stating Duality property, let's rewrite Synthesis & Analysis eqns. of CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad — (A)$$

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad — (B)$$

- Because of the symmetry b/w the above two eqns., we may say that for any transform pair, there is a dual pair with time & freq. variables interchanged.
- This is called Duality.
- Mathematically,

$$\text{If} \quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) = g(\omega)$$

$$\text{then} \quad g(t) \longleftrightarrow 2\pi x(-\omega)$$

<u>Proof:</u>

$$g(t) = \int_{-\infty}^{\infty} x(\tau) e^{-jt\tau} d\tau$$

$$g(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$\omega = t$

$\omega = -\tau$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi x(-\omega) e^{j\omega t} d\omega$$

xply & ÷de by 2π

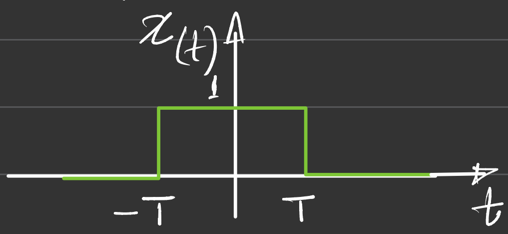
<u>Example 1</u>

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

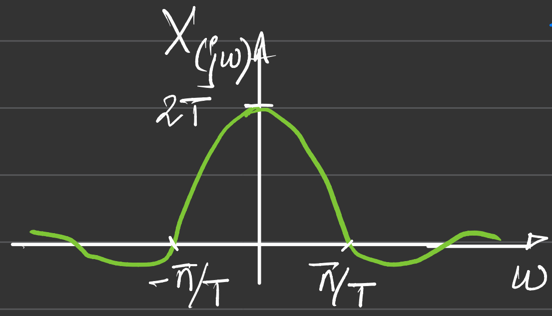
$$\Rightarrow 1 \longleftrightarrow 2\pi \delta(\omega)$$

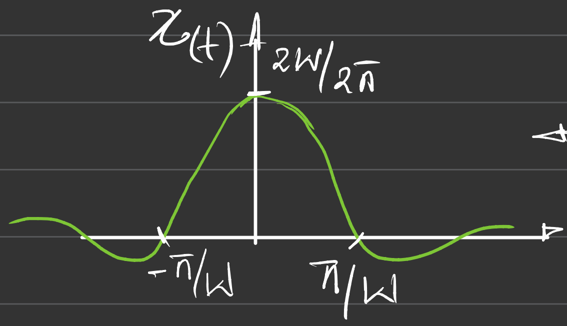
Example 4 & 5 in previous lecture

Example 2

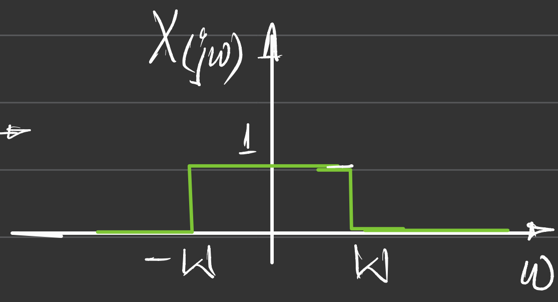


$\xleftrightarrow{\mathcal{F}}$





$\xleftrightarrow{\mathcal{F}}$



Examples 2 & 3 in the last two lectures.

Example 3

$$g(t) = \frac{4}{1+t^2} = 2\left(\frac{2}{1+t^2}\right)$$

previously we did in Example of property ④ $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2+\omega^2}$

$$x(t) = e^{-|t|} \xleftrightarrow{\mathcal{F}} X(j\omega) = \frac{2}{1+\omega^2}$$

$$G(j\omega) = ?$$

- Using Duality

$$g(t) \longleftrightarrow 4\pi e^{-|\omega|}$$

More Properties (Using Duality)

ⓐ Freq. Shifting:

$$x(t)e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(j(\omega-\omega_0))$$

ⓑ Differentiation in Freq. Domain.

$$-jt\,x(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} X(j\omega)$$

⑥ Integration in freq.-domain:

$$-\frac{1}{jt}x_{(t)} + \pi x_{(0)}\delta_{(t)} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} X_{(j\eta)}\, d\eta$$

⑦ Parseval's Relation

$$\underbrace{\int_{-\infty}^{\infty} |x_{(t)}|^2\, dt}_{\text{Energy of } x_{(t)}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_{(j\omega)}|^2\, d\omega$$

- Total Energy of the sig. may be determined either by computing the energy per unit time & integrating over all time OR by computing the energy per unit freq. & integrating over all frequencies.

Proof:

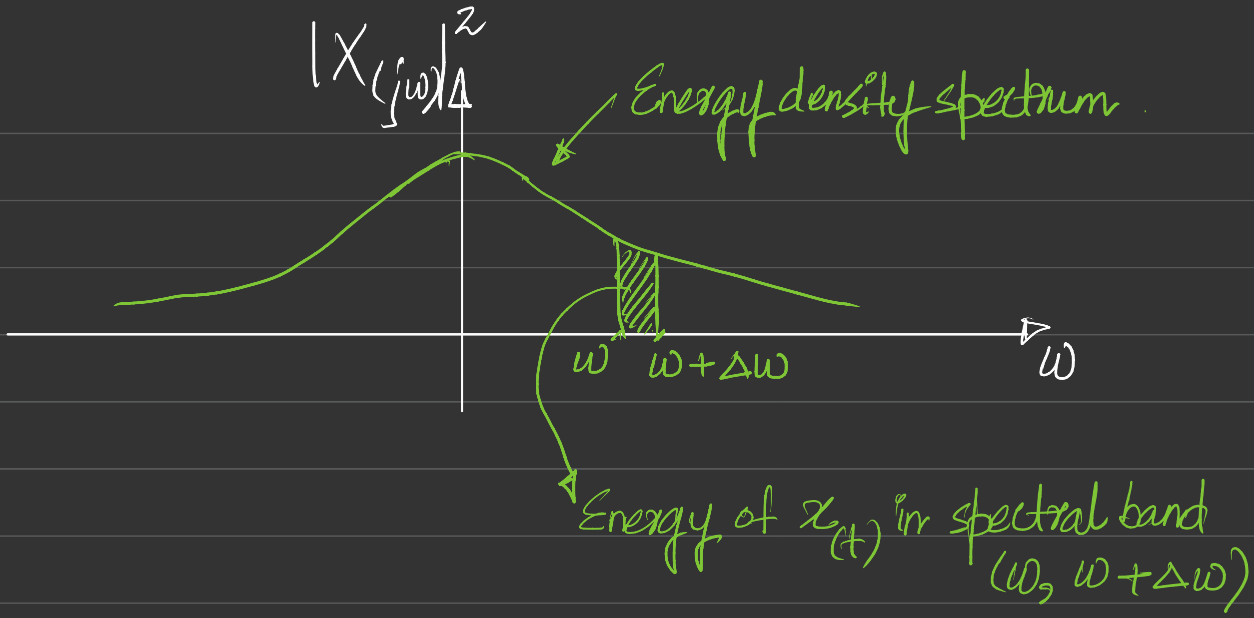
LHS $\int_{-\infty}^{\infty} |x_{(t)}|^2\, dt$

$$= \int_{-\infty}^{\infty} x_{(t)}\, x^*_{(t)}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x_{(t)}}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*_{(j\omega)}\, \underbrace{e^{-j\omega t}}\, d\omega\right) dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*_{(j\omega)}\, X_{(j\omega)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_{(j\omega)}|^2\, d\omega$$


$|X(j\omega)|^2$
Energy density spectrum
ω
$\omega+\Delta\omega$
ω
Energy of $x(t)$ in spectral band $(\omega, \omega+\Delta\omega)$